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COMMUNICATION

ON THE MAXIMUM NUMBER OF EDGES IN A HYPERGRAPH WHOSE LINEGRAPH CONTAINS NO CYCLE

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Soit $H=(X, \mathscr{E})$ un hypergraphe h-uniforme avec $|X|=n$ et soit $L_{h-1}(H)$ le graphe dont les sommets représentent les arêtes de H, deux sommets étant reliés si et seulement si les arêtes qu'ils représentent intersectent en $h-1$ sommets. Nous montrons que si $L_{h-1}(H)$ ne contient pas de cycle, alors $|\mathscr{E}| < \binom{n}{h-1}/h-1$, la borne étant exacte pour $h=2$ et pour des valeurs de n pour $h=3$. Ce problème mène à une conjecture sur les "presque systèmes de Steiner".

Let $H=(X, \mathscr{E})$ be a h-uniform hypergraph, with $|X|=n$ and let $L_{h-1}(H)$ be the graph, whose vertices are the edges of H, two vertices being joined if and only if the edges they represent intersect in $h-1$ vertices. We prove that, if $L_{h-1}(H)$ contains no cycle, then $|\mathscr{E}| < \binom{n}{h-1}/h-1$; moreover the bound is exact for $h=2$ and with some values of n for $h=3$. This problem leads to a conjecture on "almost Steiner systems".

1. Introduction

Let X be a set of n elements and let $H=(X, \mathscr{E})$ be a h-uniform hypergraph, that is $\mathscr{E}$ is a family of h-subsets of X.

We denote by $L_{h-1}(H)$ the graph, whose vertices are the edges of H and where $E_1, E_2 \in \mathscr{E}$ are joined by an edge iff $|E_1 \cap E_2| = h-1$. These graphs were previously studied (see for references [2]).

In this note we answer the following problem, which is a version of a problem due to Kachelsky and Kleitman, presented at the Problem Session of the 7th British Combinatorial Conference (Cambridge, August 1979).

Theorm 1. *Suppose $L_{h-1}(H)$ contains no cycle, then*

$$|\mathscr{E}| < \frac{\binom{n}{h-1}}{h-1}. \tag{1}$$

Moreover the bound is exact for $h = 2$ and for some values of n for $h = 3$.

2. Proof of inequality (1)

Let $F \subset X$, $|F| = h-1$. The F is contained in at most 2 different members of $\mathscr{E}$, as otherwise for $E_1, E_2, E_3 \in \mathscr{E}$ and containing F, we would have a triangle in $L_{h-1}(H)$.

Hence, for $F \subset E \in \mathscr{E}$, $|F| = h-1$ there are only two possibilities, either F is contained only in E or there is exactly one more edge E', with $F \subset E'$.

Let $a(E)$ denote the number of $(h-1)$-subsets of $E \in \mathscr{E}$, which are contained in some $E' \in \mathscr{E}$, $E' \neq E$. Then the vertex E is joined to exactly $a(E)$ other vertices of $L_{h-1}(H)$.

Let m be the number of edges of $L_{h-1}(H)$, then we obtain

$$2m = \sum_{E \in \mathscr{E}} a(E). \tag{2}$$

On the other hand, $L_{h-1}(H)$ contains no cycle, hence we also have

$$m \leq |\mathscr{E}| - 1. \tag{1}$$

Combining (2) and (3) we obtain

$$|\mathscr{E}| - 1 \geq \tfrac{1}{2} \sum_{E \in \mathscr{E}} a(E). \tag{4}$$

Let $c(H)$ be the number of $(h-1)$-subsets of X, which are contained in some edge E of H. Obviously $c(h) \leq \binom{n}{h-1}$. We obtain

$$c(H) = \sum_{E \in \mathscr{E}} (h - a(E)) + \tfrac{1}{2} \sum_{E \in \mathscr{E}} a(E) \leq \binom{n}{h-1}$$

or equivalently

$$c(H) = h\,|\mathscr{E}| - \tfrac{1}{2} \sum_{E \in \mathscr{E}} a(E) \leq \binom{h}{h-1}. \tag{5}$$

From (5) using (4) we deduce

$$h\,|\mathscr{E}| - (|\mathscr{E}| - 1) = (h-1)\,|\mathscr{E}| + 1 \leq \binom{n}{h-1}$$

or equivalently $|\mathscr{E}| < \binom{n}{h-1}/h-1$ proving (1).

Remark. Let us define $L_{\geq i}(H)$ to be the graph whose vertices are the edges of H and where $E_1 \neq E_2 \in \mathscr{E}$ are joined iff $|E_i \cap E_2| \geq i$ $(1 \leq i \leq h-1)$. Suppose that

$L_{\geq i}(H)$ contains no cycle. Then the above proof yields

$$|\mathscr{E}| < \binom{n}{i} \Big/ \left(\binom{h}{i} - 1\right).$$

3. Constructions

For $h = 2$, we can take any tree to show that the bound (1) is exact.

Let now $h = 3$.

Suppose $n = 8k + 5$ is a prime and 2 is a primitive root modulo n, that is the numbers $\pm 2^\alpha$, $\alpha = 0, 1, \ldots, \frac{1}{2}(n-3)$ are all different. (It is conjectured in number theory that there exists an infinite number of such n's.)

Let H be a hypergraph with vertex set $\{0, 1, \ldots, n-1\}$ and edge set:

$$\{E_i^\alpha = \{i, i + 2^{2\alpha}, i + 2^{2\alpha+1}\}, i = 1, \ldots, n-1;\ \alpha = 0, 1, \ldots, \tfrac{1}{4}(n-5)\}$$
$$\cup \{E_0^\alpha = \{0, 2^{2\alpha+1}, 2^{2\alpha+2}\}, \alpha = 0, 1, \ldots, \tfrac{1}{4}(n-9)\}$$

(all the numbers are taken modulo n).

We assert that $L_{h-1}(H)$ contains no cycle. To see this, observe that the subgraph generated by $X^\alpha = \{E_i^\alpha,\ i = 1, \ldots, n-1\}$ is a Hamiltonian path on X^α.

Moreover $|E^\alpha \cap E_j^{\alpha'}| \leq 1$, for $1 \leq i, j \leq n-1$, $\alpha \neq \alpha'$ as 2 is a primitive root. Therefore there are no edges in $L_{h-1}(H)$ joining X^α to $X^{\alpha'}$.

Finally the vertex E_0^α is joined in $L_{h-1}(H)$ only to $E_{2^{2\alpha+1}}^\alpha$ and to $E_{n-2^{2\alpha+2}}^{\alpha+1}$. Hence $L_{h-1}(H)$ is a tree (see Fig. 1 for an example, $n = 13$).

We have $|\mathscr{E}| = \frac{1}{4}(n-1)(n-1) + \frac{1}{4}(n-5) = \frac{1}{4}n(n-1) - 1$ which proves that the bound (1) is optimal for the mentioned values of n.

For the other values of n, we can prove:

Theorm 2. *For every n, there exists a 3-uniform hypergraph H of order n such that $L_2(H)$ has no cycle and $\frac{1}{2}\binom{n}{2} - |H| \leq \frac{1}{4}n$.*

We use the following lemma:

Lemma 1. *There exists a sequence of matchings (graphs with maximum degree 1) C_j such that*

(a) $|C_{2i-1}| = |C_{2i}| = i$,

(b) *the vertex-set of* $G_j = C_1 \cup \cdots \cup C_j$ *is* $\{x_1, \ldots, x_{j+1}\}$

(c) $C_j \cap C_k = \emptyset$ $(j \neq k)$,

(d) *the degree-sequence of* G_{2i} *is dominated by* $\{1, 2, 3, \ldots, 2i-3, 2i-2, 2i-1, 2i-1\}$, *for* $i \geq 3$.

The construction of H in Theorem 2 is the following:

$$H = \{\{x_j\} \cup E \mid E \in C_{j-2}, j = 3, \ldots, n\}.$$

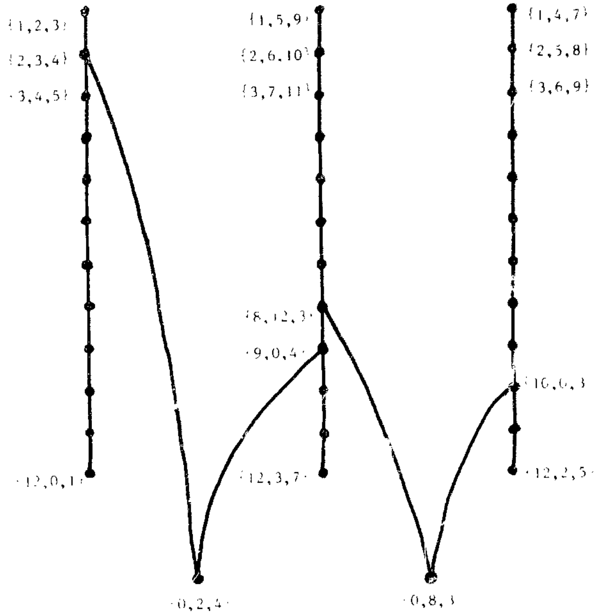

Fig. 1.

Lemma 1 is proved by induction, with the aid of the Tutte–Berge matching theorem (see for instance [1]). This method is also valid for $h>3$. However, in this case, Lemma 1 should be replaced by results not yet proved such as the following one:

Conjecture. There exists a sequence of $(h-1)$-uniform hypergraphs S_j such that:

(a) the vertex-set of $S_1 \cup \cdots \cup S_j$ is $\{x_1, \ldots, x_{j+h-2}\}$,

(b) $S_j \cap S_k = \emptyset$ $(j \neq k)$,

(c) every two edges of S_j meet in at most $h-3$ vertices,

(d) $|S_j| = \binom{j+h-2}{h-2}/(h-1) - O(j^{h-3})$.

The last two conditions mean that S_j is an "almost Steiner system". (Note $O(j^{h-3}) \leq c_h j^{h-3}$ for some constant c_h).

References

[1] C. Berge, Graphs and hypergraphs (North-Holland, Amsterdam, 1976).

[2] J.-C. Bermond, A. Germa and M.-C. Heydemann, Graphes representatifs d'hypergraphes, in: Coll. Math. Discretes, Bruxelles (1978), Cahiers du C.E.R.O. 20 (1978) 325–329.